$T_A(\vec{x}) = A\vec{x}$ $f(x) = 5x$

$T_A = A$ $f = 5$

Due Sun

4.1 – Real Vector Spaces

(as opposed to complex)

Definition: (Vector space)

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects **u** and **v** in V an object **u** + **v**, called the **sum** of **u** and **v**; by **scalar multiplication** we mean a rule for associating with each scalar k and each object **u** in V an object k**u**, called the **scalar multiple** of **u** by k. If the following axioms are satisfied by all objects **u**, **v**, and **w** in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

1. * If **u** and **v** are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V. ← closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There exists an object in V, called the **zero vector**, that is denoted by **0** and has the property that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all **u** in V.
5. For each **u** in V, there is an object $-\mathbf{u}$ in V, called a **negative** of **u** such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. * If k is any scalar and **u** is an object in V, then $k\mathbf{u}$ is in V. ← closed under scalar mult.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Notes: ① Vector add. and scalar mult. can be any rules that satisfy these axioms

② A vector is any object in a vector space

#6 Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces, Identify the vector space axioms that fail.

The set of all n-tuples of real numbers that have the form $(x,\ x,\ \ldots,\ x)$ with the standard operations on R^n.

Let $\vec{u}=(u,u,\ldots,u)$, $\vec{v}=(v,v,\ldots,v) \in \mathbb{R}^n$, $k \in \mathbb{R}$

① $\vec{u}+\vec{v}=(u+v, u+v, \ldots, u+v)$ ✓

②, ③, ⑦–⑩ are satisfied by properties of real #s (Thm 3.1.1)

⑥ $k\vec{u}=(ku, ku, \ldots, ku)$

④ $\vec{0}=(0,0,\ldots,0)$ ⑤ $-\vec{u}=(-u,-u,\ldots,-u)$

This is a vector space.

Examples of Vector Spaces

1. The simplest vector space is $\{\mathbf{0}\}$.
2. R^n with the usual operations of addition and scalar multiplication of n-tuples
3. R^∞, the set of infinite sequences of numbers with the usual operations of addition and scalar multiplication performed componentwise
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication, denoted M_{mn} $\begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 2 \end{bmatrix} \in M_2$
5. The set of real-valued functions $f(x)$ that are defined for all $x \in R$, denoted $F(-\infty, \infty)$ with addition and scalar multiplication $(\mathbf{f}+\mathbf{g})(x) = f(x)+g(x)$ and $(k\mathbf{f})(x) = kf(x)$
6. The set of polynomials of degree $\leq n$, denoted P_n

$2f(x) = 2\sin x$

$\vec{f}(x)=\sin x$ $g(x)=x^2$ $(f+g)(x)=\sin x+x^2$

Example: Show why #5 is a vector space.

Let $\vec{f}$ and $\vec{g} \in F(-\infty, \infty)$ and $k \in R$

① $(\vec{f}+\vec{g})(x) = f(x)+g(x) = (f+g)(x)$ ✓

⑥ $(k\vec{f})(x) = k f(x) = k\vec{f}(x)$ ✓

At any point x, $\vec{f}(x)$ & $\vec{g}(x)$ are real #s

thus, 2, 3, 7, 8, 9, 10 are satisfied

by properties of real #s.

④ $\vec{0}$ is the zero function $f(x)=0$ ✓

⑤ $-\vec{f}(x) = -f(x)$ ✓

Example 4.1.7: Not a vector space

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Define $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $k\mathbf{u} = (ku_1, 0)$.

Let $k=1$: $k\vec{u} = (1 \cdot u_1, 0)$

$= (u_1, 0) \neq \vec{u}$

Example 4.1.8: Unusual vector space

Let V be the set of positive real numbers, so $\mathbf{u} = u$ and $\mathbf{v} = v$ are positive real numbers. Define $\mathbf{u} + \mathbf{v} = uv$ and $k\mathbf{u} = u^k$.

vector addition is multiplication; scalar mult. is exponentiation

$\vec{u} = 5$, $\vec{v} = 2 \Rightarrow \vec{u} + \vec{v} = \vec{5} + \vec{2} = 10$

$k = 3 \rightarrow 3\vec{u} = 5^3 = 125$

The set is closed under vector add & scalar mult. ①, ⑥

vector add. is comm. & assoc because mult. of real #s is. ②, ③

The zero element: $\vec{0} = 1$ because

$\vec{0} + \vec{u} = 1u = u = \vec{u}$ ④

The negative of a vector $\vec{u}$ is its reciprocal:

$1\vec{u} = u^{-1} = \frac{1}{u}$ and $\vec{u} + (-\vec{u}) = u\left(\frac{1}{u}\right) = 1 = \vec{0}$ ⑤

7. $k(\vec{u} + \vec{v}) = (uv)^k = u^k v^k = k\vec{u} + k\vec{v}$

8. $(k+m)\vec{u} = u^{k+m} = u^k u^m = k\vec{u} + m\vec{u}$

9. $k(m\vec{u}) = (u^m)^k = u^{mk} = u^{km} = (km)\vec{u}$

10. $1\vec{u} = u^1 = u = \vec{u}$

Example: Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 2,\ u_2 + v_2)$, $k\mathbf{u} = (ku_1, ku_2)$.

a. Show that Axioms 4 and 5 hold.

b. Find all axioms that fail to hold.

Let $\vec{0} = (a, b)$. $\vec{0} + \vec{u} = (a + u_1 + 2, b + u_2) = (u_1, u_2)$

$a = -2$, $b = 0 \Rightarrow \vec{0} = (-2, 0)$ ✓

Let $-\vec{u} = (a, b)$. $\vec{u} + (-\vec{u}) = (u_1 + a + 2, u_2 + b) = (-2, 0)$

$u_1 + a + 2 = -2 \Rightarrow a = -u_1 - 4$, $b = -u_2$

so $-\vec{u} = (-u_1 - 4, -u_2)$

7. $k(\vec{u} + \vec{v}) = k(u_1 + v_1 + 2, u_2 + v_2)$

$= (ku_1 + kv_1 + 2k, ku_2 + kv_2)$

$k\vec{u} + k\vec{v} = (ku_1, ku_2) + (kv_1, kv_2)$

$= (ku_1 + kv_1 + 2, ku_2 + kv_2)$

$k(\vec{u} + \vec{v}) \neq k\vec{u} + k\vec{v}$ so (7) fails

(8) will likewise fail

Theorem 4.1.1 Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:

a) $0\mathbf{u} = \mathbf{0}$ ✓

b) $k\mathbf{0} = \mathbf{0}$

c) $(-1)\mathbf{u} = -\mathbf{u}$

d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

Example: Prove that the zero vector in any vector space is unique.

Pf: Let V be a vector space and let $\vec{0}$ be a vector such that $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\forall\ \vec{u}\in V$. Suppose $\exists$ (there is) $\vec{0}'\in V$ $\ni$ (such that) $\vec{0}'+\vec{u}=\vec{u}+\vec{0}'=\vec{u}\ \forall\ \vec{u}\in V$.

Let $\vec{u}=\vec{0}$. Then $\vec{0}'+\vec{0}=\vec{0}$

But $\vec{0}'+\vec{0}=\vec{0}'$

So $\vec{0}'=\vec{0}$. ✓

So the zero vector is unique.

Thm 4.1.1(c): If $\vec{u}\in V$ a vector space, $k\in\mathbb{R}$ and if $k\vec{u}=\vec{0}$, then $k=0$ or $\vec{u}=\vec{0}$.

Pf: Let $\vec{u}\in V$, $k\in\mathbb{R}$.

Suppose $k\vec{u}=\vec{0}$.

If $k=0$, then $0\vec{u}=\vec{0}$ done.

If $k\neq 0$, then $k\vec{u}=\vec{0} \Rightarrow \vec{u}=\frac{1}{k}\vec{0}$

then $\vec{u}=\vec{0}$.